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Rigorous Entropy-Energy Arguments

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We present a method for making rigorous various arguments which predict that certain situations are unstable because of a balance of energy vs. entropy. As applications, we give yet another proof that the two-dimensional plane rotor has no spontaneous magnetization and we make rigorous Thouless' arguments on the one-dimensional Ising model with coupling J/n^2.

KEY WORDS: Entropy; phase transitions; Ising model; plane rotor; spin systems; one-dimensional models; two-dimensional models; symmetry breaking; Thouless effect.

1. INTRODUCTION

One of the most appealing heuristic devices for understanding when a phase transition will take place is to consider the balance of energy versus entropy. Probably the simplest example is the well-known[9,27] heuristic argument for the absence of phase transitions in one-dimensional systems with short-range forces: Suppose that there were two distinct phases with the same bulk free energy. Then, starting with a single phase, we can insert a droplet of size L of the other phase: the energy cost is finite (independent of L), since the forces are short range; while the droplet can be inserted in approximately N/L places (where N is the length of the system), leading to an entropy gain on the order of $k\ln(N/L)$. Thus, for any temperature $T > 0$, the free energy change const $- kT\ln(N/L)$ is negative for sufficiently large N; that is, the putative pure phase is unstable.

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An analogous heuristic argument[8,9,10,27] leads to the expectation that the d-dimensional nearest-neighbor plane rotor has no spontaneous magnetization for $d \leqslant 2$: For suppose otherwise. Then, starting with the magnetized phase, we can flip all the spins in a block of size L; to do this without costing too much energy, we surround this block by L more shells with spins rotated by angles $\pi(1 - 1/L), \pi(1 - 2/L), \ldots$. It is easy to see that the energy cost is on the order of L^{d-2}, while the entropy gain is on the order of $k \ln(N^d/L^d)$. Thus, for $d \leqslant 2$, the putative magnetized phase is unstable at any temperature $T > 0$.[3]

In both of the above examples, the heuristic expectation has been rigorously proven by several different methods: see Refs. 23, 24, 3 for the one-dimensional case, and Refs. 18, 4, 17, 13 for the two-dimensional plane rotor. However, none of these proofs directly follows the heuristic argument. Our main goal here is to show how to estimate entropy shifts in such a way that the heuristic entropy-energy arguments—in the models discussed above as well as others—can be made rigorous. After the completion of our work, we received a preprint of Pfister[21] which is closely related and which we shall discuss shortly.

Of course, the heuristic arguments presented above suffer from somewhat vague notions of "phase" and "entropy." For instance, in the plane-rotor example one seems to be taking "magnetized phase" to mean "a configuration with all spins aligned." But this approach—considering only the ground-state configuration and selected finite-energy perturbations thereof—is suspect, since at temperature $T > 0$ it is infinite-energy configurations which are really relevant. A much better version of the entropy-energy argument is implicit in the work of Landau and Lifshitz[14] (developed further by Thouless[26]), who appeal essentially to the Gibbs variational principle: any translation-invariant equilibrium state ρ must maximize the quantity $s(\rho) - \rho(A_\Phi)$. Here $s(\rho)$ is the entropy per unit volume of the state (probability measure) ρ, and $\rho(A_\Phi)$ is the expectation value in the state ρ of the energy per unit volume associated with the interaction Φ; see Israel[11] or Simon[25] for notation, history, and precise proofs. The idea is then: given a translation-invariant state ρ with nonzero magnetization, we construct a new translation-invariant state ρ' by randomly flipping various spins; if we can show that

$$s(\rho') - \rho'(A_\Phi) > s(\rho) - \rho(A_\Phi) \tag{1.1}$$

then it follows that ρ cannot have been an equilibrium state.

[3] The alert reader may wonder why this conclusion is not also valid for $d > 2$. The answer, which will become apparent when we try to spell out this argument in complete rigor, is that the excess of entropy over energy must be shown for arbitrarily large L, with N large but not *too* large. See Sections 2–5, below.

Actually, in many of our arguments we can deal with finite-volume states only; this turns out to simplify matters considerably. Then the version of the variational principle that we need is a simple consequence of Jensen's inequality:

Theorem 1.1. Let $d\mu_0$ be a probability measure, and let f, g be positive functions with $\int f\,d\mu_0 = \int g\,d\mu_0 = 1$. Now define

$$S(f) = -\int f \ln f\,d\mu_0 \tag{1.2}$$

Then

$$S(f) + \int (\ln g) f\,d\mu_0 \leqslant 0 \tag{1.3}$$

Proof. The left-hand side of (1.3) is just

$$\int \ln(g/f) f\,d\mu_0 \leqslant \ln\Big(\int g\,d\mu_0\Big) = 0$$

where the inequality is Jensen's inequality for the probability measure $f\,d\mu_0$ and we use the concavity of the log. ■

To describe the idea of our construction, imagine picking at random one of n distinct blocks of spins and flipping all the spins in this block, leaving all other spins unaltered. One imagines this extra randomness as increasing entropy. If one were dealing with n disjoint events, the increase would be $\ln n$, i.e., if ν_i is the probability measure $f_i\,d\mu_0$ and the f_i's have disjoint supports, then

$$S\left(n^{-1}\sum_{i=1}^{n} f_i\right) = n^{-1}\sum_i S(f_i) + \ln n \tag{1.4}$$

Of course, if the f_i's are not disjoint, then one will not gain a full $\ln n$, e.g., if $\nu_1 = \cdots = \nu_n$, there is obviously no $\ln n$ in (1.4). In Section 2, we obtain lower bounds on

$$S\left(n^{-1}\sum_{1}^{n} f_i\right) - n^{-1}\sum_i S(f_i) \tag{1.5}$$

in case the f_i's are only "almost disjoint." Of course, for flipping distinct sets of spins to yield almost disjoint measure we will need to find functions F whose probability distributions are very different in the different ν_i. The natural choice for F will be the total magnetization of the block being flipped; to get sufficiently distinct probability distributions, we will need to have the magnetization in the unflipped state nonzero (and the block size sufficiently large). In this case we could hope to obtain a violation of (1.3), and thereby conclude that the magnetization in any equilibrium state must be zero.

As a warm-up to the method, we prove in Section 3 that a one-dimensional Ising model with pair interaction $J(i-j)$ obeying $\sum_i |i|\,|J(i)| < \infty$ has no spontaneous magnetization; a stronger result is originally due to Ruelle.[23] In Section 4 and the Appendix, we discuss the absence of spontaneous breaking of continuous symmetries in two dimensions. Section 5 is the most subtle of the models we consider, in that an infinite energy shift, albeit a logarithmic one, is balanced by an entropy change.

We close this introduction by comparing our method with its closest relative, one which, rather than discussing energy balanced against entropy, describes certain finite-energy situations. In its simplest version, this argument applies to situations in which a uniformly bounded energy change results from an arbitrary change of boundary conditions: this is the situation in Section 3 but not Section 4. This version was discovered by Sakai[24] and rediscovered by Bricmont *et al.*,[3] both of whom used it to recover Ruelle's result on uniqueness of state for one-dimensional systems with $\sum |i|\,|J(i)| < \infty$. Recently, Pfister[21] found a beautiful argument which applies to two-dimensional systems with continuous symmetries: while the initial steps are, of necessity, somewhat different, the final steps are identical to those in the Sakai argument.

The Pfister argument has a critical advantage over ours: it deals with *all* equilibrium states, while we are typically limited to proving that some symmetry is unbroken in *translation-invariant* equilibrium states. But our method has its own advantages: it does not require information on disintegration of equilibrium states with respect to the algebra of observables at infinity, and thus it seems to us more intuitive. Moreover, at the present moment it seems more systematic. Most importantly, we can say something about cases like those of Section 5 with infinite energy shift.

2. ENTROPY OF MIXTURES

In this section we want to compare the entropy of an average $n^{-1}\sum_{i=1}^{n} \nu_i$ with the average entropy. This "entropy of taking a statistical mixture" should be distinguished from the often-discussed "entropy of mixing" which, while related, is distinct. We define S by (1.2).

Theorem 2.1. Let $d\mu_0$ be a probability measure. Let $d\nu_i = f_i\, d\mu_0$, $i = 1, \ldots, n$ be n probability measures. Suppose that there is a $c \geqslant 0$ such that for each i there exixts a set A_i with

(i) $$\nu_i(A_i^c) \leqslant c/n$$

(ii) $$\sum_{j \neq i} \nu_i(A_j) \leqslant c$$

where A_i^c is the complement of A_i. Let $f = n^{-1}\sum_{i=1}^n f_i$. Then

$$S(f) \geqslant n^{-1}\sum_{i=1}^{n} S(f_i) + \ln n - 4c^{1/2} \tag{2.1}$$

Proof

$$\begin{aligned}
S(f) &- n^{-1}\sum_i S(f_i) - \ln n \\
&= n^{-1}\left\{ -\sum_{i=1}^{n}\int f_i \ln\left[\left(\sum f_j\right)/f_i\right] d\mu_0 \right\} \\
&= 2n^{-1}\left\{ \sum_{i=1}^{n}\int d\nu_i \left[-\ln\left(\frac{f_1+\cdots+f_n}{f_i}\right)^{1/2}\right]\right\} \\
&\geqslant 2n^{-1}\sum_{i=1}^{n}\left\{ -\ln\left[\int d\mu_0 \left(f_i^2 + \sum_{j\neq i} f_j f_i\right)^{1/2}\right]\right\} && (2.2)\\
&\geqslant 2n^{-1}\sum_{i=1}^{n}\left\{ -\ln\left[1+\int d\mu_0 \left(\sum_{j\neq i} f_j f_i\right)^{1/2}\right]\right\} && (2.3)\\
&\geqslant 2n^{-1}\sum_{i=1}^{n}\int d\mu_0 \left(\sum_{j\neq i} f_j f_i\right)^{1/2} && (2.4)
\end{aligned}$$

where (2.2) uses Jensen's inequality and the convexity of $-\ln$, (2.3) uses $(a+b)^{1/2} \leqslant a^{1/2} + b^{1/2}$, and (2.4) uses $\ln(1+x) \leqslant x$. For i fixed use the Schwarz inequality to get

$$\begin{aligned}
\int d\mu_0 \left(\sum_{j\neq i} f_i f_j\right)^{1/2} &= \int_{A_i} d\mu_0\left(\sum_{j\neq i} f_j f_i\right)^{1/2} + \int_{A_i^c} d\mu_0\left(\sum_{j\neq i} f_j f_i\right)^{1/2} \\
&\leqslant \nu_i(A_i)^{1/2}\left[\sum_{j\neq i}\nu_j(A_i)\right]^{1/2} + \nu_i(A_i^c)^{1/2}\left[\sum_{j\neq i}\nu_j(A_i^c)\right]^{1/2} \\
&\leqslant 1\cdot c^{1/2} + (c/n)^{1/2}(n-1)^{1/2} \leqslant 2c^{1/2}
\end{aligned}$$

which given (2.4) proves (2.1). ■

3. FINITE ENERGY SHIFT: ONE DIMENSION

As an illustration of our method, let us prove the following theorem.

Theorem 3.1. The one-dimensional spin-1/2 Ising ferromagnet with coupling $J(i-j) \geqslant 0$ obeying

$$\sum_{r\neq 0} |r| J(r) < \infty \tag{3.1}$$

has zero spontaneous magnetization.

Remarks. (1) By correlation inequalities,[9] the result, once proven for $J(n) \geqslant 0$, is true for all $J(r)$ with (3.1) replaced by $\sum_{r\neq 0}|rJ(r)| < \infty$.

(2) While we exploit correlation inequalities in the proof for technical ease, they can be avoided.

(3) Ruelle [23] has proven the stronger result that (3.1) implies a unique equilibrium state. This result can be proven by Sakai's method[24] (see Ref. 3), which is probably the "best" proof. In the ferromagnetic case, this also follows from Theorem 3.1 together with the theorem of Lebowitz and Martin-Löf.[16]

Proof. Let $\langle\ \rangle_{+,l}$ denote the plus-boundary-condition state in the region $i = l, -l+1, \ldots, l$; $P_{+,l}$ will denote probabilities in that state. Let $\langle\ \rangle_{+,\infty}$ denote the limiting state obtained as $l \to \infty$; correlation inequalities imply that $\langle\ \rangle_{+,\infty}$ is translation invariant and ergodic (in fact extremal, hence mixing). We must prove that

$$\langle \sigma_0 \rangle_{+,\infty} \equiv m \tag{3.2}$$

is zero, so suppose $m \neq 0$. Let $\epsilon > 0$ be given. We first claim that we can find k such that for each block B of k successive spins and each l so large that $B \subset \{-l, \ldots, l\}$, we have that

$$P_{+,l}\Big(\sum_{i\in B} \sigma_i \leqslant 0\Big) \leqslant \epsilon \tag{3.3}$$

For let

$$a = \Big\langle \sum_{i\in B} \sigma_i \Big\rangle_{+,l}$$

and

$$F = \sum_{i\in B} \sigma_i - a$$

Then

$$P_{+,l}\Big(\sum_{i\in B} \sigma_i \leqslant 0\Big) \leqslant a^{-2}\langle F^2 \rangle_{+,l} \tag{3.4}$$

But, by correlation inequalities [GKS for (3.5), GHS for (3.6)]:

$$a \geqslant mk \tag{3.5}$$

$$\big[\langle \sigma_i\sigma_j\rangle_{+,l} - \langle\sigma_i\rangle_{+,l}\langle\sigma_j\rangle_{+,l}\big] \leqslant \big[\langle \sigma_i\sigma_j\rangle_{+,\infty} - \langle\sigma_i\rangle_{+,\infty}\langle\sigma_j\rangle_{+,\infty}\big] \tag{3.6}$$

so (3.4) becomes

$$P_{+,l}\Big(\sum_{i\in B} \sigma_i \leqslant 0\Big) \leqslant m^{-2}\Big\{k^{-2}\sum_{i,j\in B}\big[\langle \sigma_i\sigma_j\rangle_{+,\infty} - \langle\sigma_i\rangle_{+,\infty}\langle\sigma_j\rangle_{+,\infty}\big]\Big\}$$

Since $m > 0$ by hypothesis, and the quantity in brackets goes to zero as $k \to \infty$ by ergodicity of $\langle\ \rangle_{+,\infty}$, we have (3.3).

Given n, first pick k so large that (3.3) holds with $\epsilon = 1/n$, and then pick l so that $2l + 1 \geqslant nk$. In $\{-l, \ldots, l\}$, pick n disjoint blocks of k successive spins each; call them $B_1, \ldots, B_n$. We will obtain a contradiction with Theorem 1.1 by starting with the Gibbs state $\langle \cdot \rangle_{+,l}$ and flipping "at random" one of the blocks B_i. Explicitly, in (1.3), let $d\mu_0$ be the uncoupled Ising measure, and let g be the Gibbs density for $\langle \cdot \rangle_{+,l}$ with respect to $d\mu_0$, i.e.,

$$\ln g = (\text{const}) + \sum_{-l \leqslant i < j \leqslant l} J(i-j)\sigma_i\sigma_j + \sum_{\substack{|i| > l \\ |j| \leqslant l}} J(i-j)\sigma_j \tag{3.7}$$

Let f_i $(1 \leqslant i \leqslant n)$ be the function obtained from g by flipping the spins in block B_i, and let $d\nu_i = f_i\, d\mu_0$. By the symmetry of $d\mu_0$, we have

$$S(f_i) = S(g) \tag{3.8}$$

Let $A_i = \{\sigma \mid \sum_{j \in B_i} \sigma_j \leqslant 0\}$. By (3.3) with the assumed value $\epsilon = 1/n$, conditions (i) and (ii) of Theorem 2.1 hold (with $c = 1$), so if $f = n^{-1}\sum_{i=1}^{n} f_i$, then by (2.1) and (3.8),

$$S(f) \geqslant S(g) + \ln n - 4 \tag{3.9}$$

On the other hand, (3.7) and the elementary estimate

$$\sum_{\substack{i \geqslant 0 \\ j < 0}} J(i-j) = \sum_{r > 0} rJ(r) = \frac{1}{2} \sum_{r \neq 0} |r| J(r) \tag{3.10}$$

shows that the energy shift is

$$\Delta E \equiv \int (g - f) \ln g\, d\mu_0 \leqslant 2 \sum_{r \neq 0} |r| J(r) \tag{3.11}$$

[Notice, by the way, that it would be no good to use

$$\sum_{\substack{i \geqslant 0 \\ -k \leqslant j < 0}} J(i-j) \leqslant k \sum_{r > 0} J(r) \tag{3.12}$$

in place of (3.10), since k is n-dependent and there would be no guarantee that $k\sum_{r>0} J(r)$ could be dominated by $\ln n$; indeed, k must grow at least as fast as n.]

Combining (3.9) and (3.11), we conclude that

$$S(f) + \int f \ln g\, d\mu_0 \geqslant \ln n - 4 - 2 \sum_{r \neq 0} |r| J(r) \tag{3.13}$$

For large n, this contradicts Theorem 1.1; so we conclude that $m = 0$ since we obtain a contradiction if $m \neq 0$. ∎

To summarize, we have proven that if the infinite-volume magnetization were nonzero, then we could introduce enough extra randomness into a finite-volume Gibbs state so that the entropy gain more than compensates the energy gain, violating the fact that the Gibbs state is an absolute maximum in the Gibbs variational principle.

4. FINITE ENERGY SHIFT: TWO DIMENSIONS

Our goal in this section is to prove the following theorem.

Theorem 4.1. The spontaneous magnetization is zero in a two-dimensional ferromagnetic plane rotor model with pair coupling $J(i-j)$ obeying

$$\sum_{i\neq 0} |i|^2 J(i) < \infty \tag{4.1}$$

Remarks. (1) In the Appendix we shall extend this result in several ways (although we will strengthen the hypothesis to require J be finite range): we will avoid correlation inequalities and thereby deal with much more general "spins" than plane rotors and we will prove that every translation-invariant state has the continuous rotational symmetry.

(2) For earlier results of this genre, see Refs. 18, 4, 17, 13, and 21. Pfister[21] proves a strictly stronger result.

Proof. We emphasize those steps which differ from those in Section 3. We let $\langle\ \rangle_{+,l}$ stand for the states in a box of size $(2l+1)\times(2l+1)$ centered at $(0,0)$ with all spins outside the box fixed at angle $\theta=0$. We let $P_{+,l}$ stand for the corresponding probabilities and define

$$m = \langle \sigma_i \rangle_{+,\infty}$$

Since we want to show that $m=0$, we suppose that $m\neq 0$ and will derive a contradiction. We claim that for any $\epsilon>0$, we can find k so that for any $k\times k$ block B of spins, there is an l_0 so that

$$P_{+,l}\Big(\sum_{i\in B} \sigma_i^{(1)} < 0\Big) \leqslant \epsilon \tag{4.2}$$

for $l \geqslant l_0$, where $\sigma_i^{(1)}, \sigma_i^{(2)}$ are the two components of the spin σ_i. Notice that our claim is slightly weaker than the analogous claim in Section 3 in that l_0 can be B dependent.

The proof of the above claim is basically the same as that of the analogous claim in Section 3 except that (3.6) might fail since we no longer have a GHS inequality. However, since $\langle\cdot\rangle_{+,\infty}$ is ergodic (in fact, mixing),[2,19] we can still arrange for

$$k^{-4} \sum_{i,j \in B} \left[\langle \sigma_i \sigma_j \rangle_{+,l} - \langle \sigma_i \rangle_{+,l} \langle \sigma_j \rangle_{+,l} \right] \equiv h(l)$$

to be arbitrarily small for $l = \infty$ and then can use the fact that for k, B fixed $h(l) \to h(\infty)$ as $l \to \infty$.

Now given n, let $\epsilon = 1/n$ and pick k so that (4.2) holds with this value of ϵ. Choose n disjoint $3k \times 3k$ blocks $\tilde{B}_1, \ldots, \tilde{B}_n$ and then choose l_0 so that

$$P_{+,l}\left(\sum_{j \in B_i} \sigma_j^{(1)} \leqslant 0 \right) \leqslant 1/n \tag{4.3}$$

for $1 \leqslant i \leqslant n$ and $l \geqslant l_0$, where B_i is the $k \times k$ block concentric with $\tilde{B}_i$. Let $\nu_i^{(l)}$ be the state obtained from $\langle\ \rangle_{+,l}$ by rotating the spins in $\tilde{B}_i$ as follows: spins in B_i are rotated by π, those in the square of neighbors of B_i are rotated by $\pi(1 - k^{-1})$, those in the next square by $\pi(1 - 2k^{-1}), \ldots,$ and those on the border of $\tilde{B}_i$ by π/k. Let $\nu = n^{-1} \sum_{i=1}^{n} \nu_i$ be the state where one picks one block at random and "rotates" it. Let f be the density of ν, and g the density of $\langle\ \rangle_{+,l}$. Then, by Theorem 2.1 and (4.3)

$$S(f) \geqslant S(g) + \ln n - 4 \tag{4.4}$$

The energy shift

$$\Delta E \equiv \int (g - f) \ln g \, d\mu_0$$

has two terms: interactions of spins in $\tilde{B}_i$ with the exterior of the $(2l+1) \times (2l+1)$ region and interactions within this region. By taking l large we can arrange that the former term is arbitrarily small. The latter term is a sum of terms of the form $\langle (R_i \sigma_i) \cdot (R_j \sigma_j) \rangle_{+,l} - \langle \sigma_i \cdot \sigma_j \rangle_{+,l}$, where R_i and R_j are rotations. Obviously

$$\langle (R_i \sigma_i) \cdot (R_j \sigma_j) \rangle_{+,l} = \langle \left[\left(R_j^{-1} R_i \right) \sigma_i \right] \cdot \sigma_j \rangle_{+,l}$$

Moreover,

$$\langle (R\sigma_i) \cdot \sigma_j \rangle_{+,l} = \langle \cos(\theta + \varphi_i - \varphi_j) \rangle_{+,l} = (\cos\theta) \langle \cos(\varphi_i - \varphi_j) \rangle_{+,l} \tag{4.5}$$

since $\langle \sin(\varphi_i - \varphi_j) \rangle_{+,l} = 0$ by the symmetry of $\langle\ \rangle_{+,l}$. (Here $\sigma_i^{(1)} = \cos\varphi_i$, $\sigma_i^{(2)} = \sin\varphi_i$, and θ is the angle associated with the rotation R.) Thus

$$|\langle (R\sigma_i) \cdot \sigma_j \rangle_{+,l} - \langle \sigma_i \sigma_j \rangle_{+,l}| \leqslant \tfrac{1}{2} \theta^2$$

and so

$$|\langle (R_i \sigma_i) \cdot (R_j \sigma_j) \rangle_{+,l} - \langle \sigma_i \sigma_j \rangle_{+,l}| \leqslant \tfrac{1}{2} (\theta_i - \theta_j)^2 \tag{4.6}$$

Using (4.6) and the hypothesis (4.1), a tedious but straightforward geometric argument shows (as usual[8–10,21,27]) that

$$\Delta E \leqslant \text{const} \tag{4.7}$$

independent of k (and l). From (4.4) and (4.7), one derives a contradiction to Theorem 1.1 as in Section 3. ■

5. THE THOULESS EFFECT

Our method shows its mettle in the discussion of one-dimensional models with coupling near to or precisely $J(n) = Cn^{-2}$ ($n \neq 0$). Based on the work of Thouless[26] and Anderson *et al.*,[1] one expects the following:

(i) If the falloff is strictly slower than n^{-2}, in the sense that

$$\lim_{n\to\infty} (\ln n)^{-1}\left[\sum_{j=1}^{n} j|J(j)|\right] = 0 \tag{5.1}$$

then there is no spontaneous magnetization at any temperature.

(ii) If $J(n) \geqslant Cn^{-2}$ with $C > 0$, then there is spontaneous magnetization at low temperatures.

(iii) If $J(n) = Cn^{-2}$ with $C > 0$, there is a discontinuous jump in the magnetization from zero to a strictly positive value as the temperature is decreased (the "Thouless effect").

So far, none of these expectations has been rigorously proven. However, Rogers and Thompson[22] has quite recently proven the analog of (i) with $\ln n$ replaced by $(\ln n)^{1/2}$; weaker results of this genre were found earlier by Ruelle[23] (see also Section 3) and Dyson.[5,7] Similarly, Dyson[7] has proven the analog of (ii) with n^{-2} replaced by $n^{-2}\ln\ln(n+3)$; this result was rediscovered by Kolomytsev and Rokhlenko.[12] In addition, Dyson[7] has proven (iii) for a closely related model (the hierarchical model).

We will not prove any of (i)–(iii) completely, but we will establish the following results related to (i) and (iii):

Theorem 5.1. Consider a one-dimensional spin-1/2 ferromagnetic Ising model with pair coupling obeying (5.1). If

$$\langle\sigma_i\sigma_j\rangle_{+,\infty} - \langle\sigma_i\rangle_{+,\infty}\langle\sigma_j\rangle_{+,\infty} \leqslant C[|i-j|+1]^{-a}$$

for some C, $a > 0$, then the spontaneous magnetization $m \equiv \langle\sigma_i\rangle_{+,\infty}$ is zero.

Theorem 5.2. Consider a one-dimensional spin-1/2 Ising model with pair coupling $J(n) = Cn^{-2}$ ($n \neq 0$) (absorb β into C). Suppose that

$$\chi \equiv \sum_i \left[\langle\sigma_i\sigma_0\rangle_{+,\infty} - \langle\sigma_i\rangle_{+,\infty}\langle\sigma_0\rangle_{+,\infty}\right] < \infty \tag{5.2}$$

Then either $m = 0$ or

$$m \geqslant (4C)^{-1/2} \tag{5.3}$$

Theorem 5.3. Let $m(\beta)$ be the $\langle\ \rangle_{+,\infty}$ magnetization for a one-dimensional spin-1/2 Ising ferromagnet with $J(n) = \beta n^{-2}$, and let

$$-a(\beta) = \limsup_{i\to\infty} \left\{ \ln\left[\langle \sigma_i \sigma_0 \rangle_{+,\infty} - m(\beta)^2 \right] / \ln |i| \right\}$$

Then, for each β, either $m(\beta) = 0$ or

$$m(\beta) \geqslant (4\beta)^{-1/2} \min[1, a(\beta)] \tag{5.4}$$

In particular, if $m(\beta) > 0$ for $\beta > \beta_c$ and $\lim_{\beta\downarrow\beta_c} m(\beta) = 0$, then $\lim_{\beta\downarrow\beta_c} a(\beta) = 0$.

Remarks. (1) We emphasize that the basic strategy of this section, indeed that of the whole paper, is motivated by a paper of Thouless[26]; our goal was to make his argument rigorous. Thouless even makes the distinction between Theorem 5.2 and Theorem 5.3 in that he emphasizes that his basic result (5.3) assumes "normal" fluctuations (i.e., variance going as $N^{1/2}$ with N the number of spins) of the total magnetization.

(2) The inequality (5.3) is identical to that in Thouless[26]; however, the reasoning differs somewhat in that at two different places he has extra factors, one of $1/2$ and one of 2, which cancel. Alternate versions of Thouless' argument, due to Dyson[6] and Ojo,[20] yield the stronger conclusion $m \geqslant (2C)^{-1/2}$; this might be susceptible to proof, as we shall now explain. Thouless only claims an entropy gain of $1/2 \ln N$ where we get $\ln N$ because he forces himself to look at restricted configurations (we emphasize that the correct thing is to look at states, *not* configurations). However, Thouless only gets half the energy shift we do because he flips *all* spins to the left of some point rather than a block of just n spins; he then looks only at the energy on one side of the block rather than on both sides as we do. It is not clear to us that this is legitimate since it ignores the interaction energy between the flipped spins and the boundary-condition spins; the latter spins cannot simply be ignored, since they are needed to produce a magnetization. Nevertheless, since we know for most temperatures the zero-boundary-condition state is $(\langle \cdot \rangle_{+,\infty} + \langle \cdot \rangle_{-,\infty})/2$,[15] it might be possible to improve (5.3) to read $m \geqslant (2C)^{-1/2}$.

Proof of Theorem 5.2. Consider $\langle\ \rangle_{+,(n),l}$, the plus-boundary-condition state with $4l + n^2$ spins viewed as n blocks, $B_1, \ldots, B_n$, of n spins each with $2l$ extra spins at each end. As in Section 3, using GKS and

GHS inequalities, we see that if $m \neq 0$

$$P_{+,(n),l}\left(\sum_{j \in B_i} \sigma_j < 0\right) \leqslant n^{-1}(\chi/m^2) \tag{5.5}$$

Thus, if f is the density obtained from g, the Gibbs density, by randomly flipping one block of spins, we have that

$$S(f) \geqslant S(g) + \ln n - 4\chi^{1/2}m^{-1} \tag{5.6}$$

on account of (5.5) and Theorem 2.1. On the other hand, the energy shift is

$$\Delta E = n^{-1} \sum_{i=1}^{n} \Delta E_i$$

with

$$\Delta E_i = 2C \sum_{\substack{j \in B_i \\ k \notin B_i}} \langle \sigma_j \sigma_k \rangle_{+,(n),l} |j-k|^{-2} \tag{5.7}$$

Now let $A_\pm$ be blocks of l spins at each edge of $B \equiv \cup B_i$, and let

$$m_{n,l} = \sup_{j \in A_+ \cup A_- \cup B} \langle \sigma_j \rangle_{+,(n),l}$$

It is easy to see that

$$\lim_{l \to \infty} m_{n,l} = m \tag{5.8}$$

uniformly in n. Moreover, by GHS inequalities,

$$\langle \sigma_j \sigma_k \rangle_{+,(n),l} \leqslant m_{n,l}^2 + \left[\langle \sigma_j \sigma_k \rangle_{+,\infty} - m^2 \right]$$

if $j,k \in A_+ \cup A_- \cup B$. The quantity in square brackets goes to zero as $|j-k| \to \infty$ since $\langle\ \rangle_{+,\infty}$ is mixing, and thus for any $\epsilon > 0$,

$$\Delta E_i \leqslant C_{1,\epsilon} + C_2 n l^{-1} + 2C \sum_{\substack{j \in B_i \\ k \notin B_i}} (m_{n,l}^2 + \epsilon)|j-k|^{-2}$$

The $C_{1,\epsilon}$ term is from the square bracket contributions in the region where the square bracket is larger than ϵ; note that $C_{1,\epsilon}$ is uniform in n, since only sites $j \in B_i$ within a fixed distance of the boundary of B_i contribute here. The C_2 term is from contributions with $k \notin A_+ \cup A_- \cup B$. Taking $l \geqslant n$, we see that

$$\Delta E \leqslant C_{3,\epsilon} + 4C(m_{n,l}^2 + \epsilon)\ln n$$

Using (5.8), we see that to avoid a contradiction to Theorem 1.1, we need that

$$4Cm^2 \geqslant 1 \quad \blacksquare$$

Proof of Theorem 5.3. We basically follow the above proof but now we take n blocks of k spins each. (5.5) is now replaced by

$$\mathrm{Prob} \leqslant (mk)^{-2} \sum_{0 \leqslant i,j \leqslant k} \left[\langle \sigma_i \sigma_j \rangle - m^2 \right]$$
$$\sim k^{-a(\beta)}$$

if $a < 1$, so we need to take $k \sim n^{1/a(\beta)}$ to gain $\ln n$ in entropy. The coefficient of $\ln n$ in the energy change is then $4Cm^2/a(\beta)$. For this to be larger than 1, (5.4) must hold. ■

Proof of Theorem 5.1. If we mimic the proof of Theorem 5.3, the condition (5.1) implies that the energy change cannot compensate for the $\ln n$ entropy increase. ■

APPENDIX. A RESULT OF DOBRUSHIN–SHLOSMAN TYPE

We want to use the basic ideas of this paper to prove the following result:

Theorem A.1. Consider a two-dimensional model with arbitrary compact state space Ω at each site. Consider a finite-range interaction. Suppose that a connected compact Lie group G acts on Ω such that (i) the interaction is invariant under applying the same group element to each spin; and (ii) the interaction is twice-continuously-differentiable in the group parameters under rotations of different spins by different group elements. Then every translation-invariant equilibrium state is invariant under the global group action.

Remarks. (1) This is strictly weaker than the result of Dobrushin and Shlosman[4] since they prove that every equilibrium state, whether translation invariant or not, is invariant under the global group action; Pfister[21] also proves this and allows interactions satisfying a condition slightly weaker than (4.1).

(2) Our reason for including this result is to illustrate that the use made above of correlation inequalities is not really necessary. The restriction to finite-range interactions [instead of (4.1)] is probably not essential; but in view of the beautiful proof of Pfister,[21] it hardly seems worthwhile to bore both ourselves and the reader trying to rederive this result by our methods.

Proof. We begin with a few preliminary remarks. First, since G is generated by its subgroups which are isomorphic to $U(1)$, we can suppose without loss that $G = U(1)$, i.e., label it by a number $\theta \in [0, 2\pi)$. Secondly, we need only prove that ergodic states are G invariant since every state is a limit of sums of such states. Thirdly, if ρ is the state, R_θ global rotation, and F a function of the spins, we need only show that $\rho(F \circ R_\theta) = \rho(F)$ for functions F of finitely many spins. By passing to Fourier coefficients, we can suppose that

$$F \circ R_{\theta_0} = -F \tag{A.1}$$

for some fixed θ_0 and that

$$\rho(F) = m > 0 \tag{A.2}$$

or else $\rho(F)$ will be rotation invariant.

Pick k so that F is a function of spins in a $k \times k$ array and so that k is larger than the range of the interaction. Cover Z^2 by nonoverlapping $k \times k$ blocks b_α and let F_α be the translate of F to block b_α. By (A.2) and the ergodicity of ρ, one can, given ϵ, find l so that with B an $l \times l$ block of b_α blocks (i.e., $kl \times kl$ spins) we have that

$$P_\rho\Big(\sum_{\alpha \in B} F_\alpha \leqslant 0\Big) \leqslant \epsilon \tag{A.3}$$

Let $\tilde{B}$ be an $3l \times 3l$ block concentric with B, let $\epsilon = 1/n$, and pick a cube C so large that it contains n disjoint translates $\tilde{B}_1, \ldots, \tilde{B}_n$ of $\tilde{B}$ each at least a distance k from the exterior of C. Let g be the density of the restriction of ρ to C. Since the interaction has range k we have that, for each $\tilde{B}_i$,

$$\ln g = H_{\tilde{B}_i} + R_{\tilde{B}_i} \tag{A.4}$$

where H_B is the basic interaction of spins in B with all other spins and R_B is independent of spins in B (but is generally a very complicated function due to averaging over configurations of exterior spins).

As usual, let $f_1, \ldots, f_n$ be obtained from g by rotating spins in B_i by θ_0, those in the next shell of b_α blocks by $(l-1)\theta_0/l$, those in the next shell by $(l-2)\theta_0/l, \ldots$. Let $f = n^{-1}\sum_i^n f_i$. By Theorem 2.1 and (A.3) we have

$$S(f) \geqslant S(g) + \ln n - 4$$

By (A.4), we need only show that the change in $\rho(H_{\tilde{B}_i})$ under the above rotation is bounded independently of n. Imagine replacing the above angles $\theta_0, \theta_0(1-l^{-1}), \ldots$ by arbitrary angles $\theta_1, \theta_2, \ldots, \theta_l$. Since the interactions are rotation invariant and k is the range, the energy change is a function

$$\Delta E = G(\theta_1 - \theta_2, \theta_2 - \theta_3, \ldots)$$

By hypothesis, G is C^2 and by counting up the number of spins involved, second derivatives with respect to each of the l variables are bounded by l times a constant.

Changing a single $\theta_i - \theta_{i+1}$ from zero corresponds to the change of energy for a family of states with the same entropy. Thus by (1.3), all the first-order changes must be zero. For $\theta_i = (1 - i/l)\theta_0$, $\theta_i - \theta_{i+1} = \theta_0/l$, so since the first derivatives are all zero, $\Delta E = [l]l^{-2}[lC]$ where the first factor is the number of variables, the next is $(\Delta\theta)^2$, and the last is the above C^2 norm. Thus ΔE is bounded independently of n, so (A.2) leads to a contradiction. ■

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